MIGSAA Courses 2017/18

Note: It is not permitted to take a course with the same content more than once for credit.

Standard courses -- Semester 1

Applied Analysis and PDEs 1: Dynamical Systems and Conservation Laws
Applied Mathematics Methods 1: Asymptotic and Analytical Methods
Mathematical Models 1: Continuum Mechanics
Geometry and Topology 1: Algebraic Topology
Probability 1: Foundations of Probability
Pure Analysis 1: Measure and Integration
Statistics 1: Regression and Simulation Methods
Algebra 1: Groups, Rings and Modules

Standard courses -- Semester 2

Applied Analysis and PDEs 2: Elliptic and Parabolic PDEs
Applied Mathematics Methods 2: Numerical Methods
Mathematical Models 2: Mathematical Biology and Physiology
Geometry and Topology 2: Manifolds
Probability 2: Stochastic Processes
Pure Analysis 2: Functional Analysis
Statistics 2: Modern Regression and Bayesian Methods
Algebra 2: Algebras and Representation Theory

For more detail on the content of the above courses, which are delivered by SMSTC, visit www.smstc.ac.uk

Advanced courses -- Semester 1 (These courses will be video-linked by SMSTC unless explicitly stated otherwise)

Advanced PDE I: Elliptic and parabolic PDE -- Daniel Coutand and Aram Karakhanyan (Mondays, 9.00 am - 10.30 am weeks 1 – 5, then Mondays 10.00 – 11.30 am weeks 6 – 10)

Credits: 15 credits

Brief description:
1) Examples of elliptic equations, maximum principles (strong, weak), Hopf's Lemma, comparison principle.
2) Classical solutions, Bernstein estimate, applications.
3) Schauder estimates
4) Approximation by smooth functions, Sobolev spaces, embeddings, traces
5) Weak solutions, Lax-Milgram
6) Interior regularity, Boundary regularity
7) Parabolic equations, main examples, maximum principle
8) Parabolic setting and Sobolev spaces
9) Global in time solutions for nonlinear parabolic problems with small initial data
10) Energy estimates

Prerequisites:
1) rigorous multivariable calculus (continuity, differentiability, chain rule, integration)
2) Metric spaces, Banach spaces, Hilbert space, weak/strong convergence
3) vector calculus, Green's formula, (normal, tangent/vectors, parametrisation of surfaces and curves.)

We propose 4 (2+2) sets of homework. The suggested text book is L.C. Evans, Partial Differential Equations, AMS Graduate Studies in Mathematics.

Analysis and Numerics of Stochastic PDEs - Istvan Gyongy (Tuesdays 11.00 am – 12.50 pm)

Semester 1, 2 hours over 10 weeks
Credits: 15 credits
Assessment: solving assignment problems

In the first part of the lectures basic methods of solving stochastic partial differential equations (SPDEs) of parabolic type will be presented. In particular, main results of the L_2 and L_p theories for SPDEs given in the whole Euclidean space will be summarised, and the theory of SPDEs given on domains will be presented in more details.
In the second part of the lectures methods for solving SPDEs numerically will be studied, theorems on accuracy of numerical schemes will be proved.

Applications from population genetics and stochastic filtering will be discussed.

Syllabus:

I. Main results on solvability of linear SPDEs
   • Stochastic processes with values in Sobolev spaces, and Itô formulas for their functions
   • Existence and uniqueness theorems in Sobolev spaces for SPDEs in the whole Euclidean space.
   • Solvability in weighted Sobolev spaces of SPDEs on bounded domains
   • Stochastic Fubini theorem and Itô-Wentzell formula. Feynman-Kac formulas for PDEs and SPDEs
   • SPDEs in population genetics

II. Numerical schemes for PDEs and SPDEs of parabolic type
   • Spatial discretisation, rate of convergence, accelerated schemes
   • Accuracy of implicit and explicit time discretisation
   • Fully discretised schemes
   • Localisation error
   • Splitting up approximations, accelerated splitting up schemes
   • Wong-Zakai approximations for SPDEs

A Fourier Based Approach of (stochastic) Integration and Applications - Goncalo dos Reis (Thursdays 9.00 am – 10.15 am)

Credits: 15

Brief description: In 1961, Ciesielski established a remarkable isomorphism of spaces of Hölder continuous functions and Banach spaces of real valued sequences. The isomorphism can be established along Fourier type expansions of Hölder continuous functions using the Haar-Schauder wavelet. We will start with Schauder
representations for a pathwise approach of the integral of one function with respect to another one, using Ciesielski's isomorphism. We cover the paradigm of Young integral and the rough paths integral of T. Lyons. Our approach allows understanding this more involved theory of integration, purely from an analytical perspective using Paley-Littlewood decompositions of distributions, and Bony paraproducts in Besov spaces.

We apply the theory within a probabilistic framework and express Brownian motion in this language to derive several of its properties. Moreover, the 2nd part of the course focuses on themes relating to the applications of the theory developed in the 1st part and adapts to the audience. We cover the theory of Large Deviations Principles and some of its applications to in the space of continuous functions, and we apply it to solve stochastic differential equations in a pathwise manner.

**Prerequisites:** SMSTC Probability 2 - Stochastic Processes (or “better”)

**Course duration:** 12 – 15 hours (8-10 lectures of 1h30), to take place in S1 2017/18.

Engagement with the course from students:
The lectures notes is a mix of several sources and students will be asked to type some parts of the notes. Students will present some material in class, for example, proofs of some results or solve useful examples. Roughly 2/3 of the course is delivered through standard lectures while the remaining 1/3 is based on active student participation (reading-group style).

**Syllabus.**
The lectures notes are made of a mix of several sources. A list of possible topics is as follows (although we would like to stress that the course is designed to take into account the interest of the audience, therefore the list below is to be intended as a guideline).

Ciesielski’s isomorphism and Fourier type expansions of (rough) Hölder continuous functions by means of the Haar-Schauder wavelet; isomorphism; function spaces; sequence spaces; General theory of Large deviations principles (LDP): concepts & basic properties; constructions of LDP from exponential rates of elementary sets; Contraction principle.

- Large Deviations for Brownian motion: LDPs Gaussian random variables; LDP for Brownian motion in Holder spaces; Freidling-Wentzel theory
- If time allows: we discuss further the foundations of (controlled) rough-paths theory and discuss pathwise stochastic integration and Functional Ito Calculus.

**References**
Notes by Goncalo dos Reis


**Geometric Multilinear Inequalities in Harmonic - Anthony Carbery** (Thursdays 10.30 am – 12.30 pm) **Not available to SMSTC**

Credits: 15

10 x 2 hour session

Multilinear Kakeya inequalities and multilinear restriction inequalities for the Fourier transform have played a central role in harmonic analysis over the last 10 years, underpinning such developments as
the so-called “decoupling theory” of Demeter, Bourgain and Guth, and the Bourgain–Guth approach to restriction problems. These techniques have a wide variety of applications in PDE and Number Theory (for example to Vinogradov’s Mean Value Theorem). The purpose of the course is to give an essentially self-contained introduction to this important topic. We will also consider discrete analogues concerning joints and multijoints. Some of the topics to be discussed will be:

- The Loomis–Whitney inequality
- Brascamp–Lieb inequalities
- Kakeya problems and multilinear Kakeya problems
- Kakeya problems over finite fields and Dvir’s theorem – introduction to the polynomial method
- Multilinear duality and factorisation
- Guth’s visibility theorem
- Multilinear Kakeya and Brascamp–Lieb–Kakeya theorems
- Multilinear restriction theorems
- Nonlinear Loomis–Whitney and variants
- Discrete analogues – joints and multijoints
- Applications

An important feature of the course is that it will involve the students making active contributions to its progress, by conducting mini-explorations related to some of the topics discussed, and then giving self-contained presentations during the class. It is intended that this will lead to a broader and deeper overall understanding for all involved.

Some of the topics that would be suitable for such explorations would be:

- Loomis–Whitney and induction on scales and/or the tensor power technique
- Loomis–Whitney, Gagliardo–Nirenberg, Sobolev embedding and isoperimetric inequalities
- Other applications of the polynomial method
- The Borsuk–Ulam theorem
- Geometric properties of Zhang’s invariant of a k-tuple of hypersurfaces
- Rudiments of convex geometry
- B’ezout’s theorem
- Factorisation and functional analysis
- Cheeger’s isoperimetric inequality
- Applications to combinatorics: the Bollobas — Thomason box theorem
- Guth’s visibility theorem in other settings
- Weak form of multilinear Kakeya via induction on scales
- Further applications

Prerequisites. Basic real variables and functional analysis. Basic harmonic analysis. Facility with $L^p$ spaces and basic inequalities. Familiarity with the notion of maximal functions would be helpful but not essential. We do not require the Calderón–Zygmund theory of singular integral operators.

The course is not designed for Year 1 students. However, a suitably qualified student in Year 1 could take it subject to the danger of the ‘C’ grade not being available.

Assessment. Assessment will be light-touch, qualitative and based upon students’ active participation in the course, on the scale A, B, U.

Further Notes. It is intended that this be a 20-hour course attracting 15 credits subject to full participation. Full participation means delivering one or more exploration presentation as described above – by agreement with the instructor – at a satisfactory level and playing an active role in in-class discussions throughout the duration of the course. There will be no formal tutorial support but issues raised will be discussed informally in and outside the class. It is not planned to videolink the course elsewhere in Scotland but there could be interest in England (Birmingham, King’s College).

Anthony Carbery. E-mail address: A.Carbery@ed.ac.uk
Advanced courses -- Semester 2 (These courses will be video-linked by SMSTC unless explicitly stated otherwise)

**Dispersive equations - Tadahiro Oh** *(Mondays and Wednesdays 11.00 am – 12.30 pm)*

**Not available to SMSTC**

**Credits: 15**

In this course, we study the so-called dispersive partial differential equations (PDEs) from the analytical point of view. In particular, we use harmonic analysis as our main tool and study well-posedness (existence, uniqueness, and stability under perturbation) along with the long time behaviour of solutions.

Basic topics:
- review on Lebesgue spaces: H’older’s inequality, Minkowski’s (integral) inequality, interpolation inequality
- Fourier transform: Plancherel identity, Hausdorff-Young’s inequality
- convolution: Young’s inequality, duality of products and convolutions under Fourier transform
- (fractional) Sobolev spaces: Sobolev embedding theorem via Fourier transform, algebra property of Sobolev spaces
- space-time function spaces
- nonlinear Schrödinger equations
- local well-posedness (I) on $\mathbb{R}^d$ and $\mathbb{T}^d$: via Sobolev embedding and Banach fixed point theorem
- linear solutions: dispersive estimate, Strichartz estimates (for Schrödinger equations and wave equations)
- local well-posedness (II) on $\mathbb{R}^d$: via Strichartz estimates
  (II.a) subcritical regularity
  (II.b) critical regularity
- conservation laws, global existence in $L^2$ and $H^1$
- virial identity, finite-time blowup solutions
- basic scattering theory

Some advanced topics (to be covered, depending on a choice of an instructor):
- local well-posedness (III) in the periodic setting: periodic Strichartz estimates, Fourier restriction norm method
- energy method: energy estimate, parabolic regularization
- local smoothing estimates, maximal function estimates
- $I$-method/acceleration law
- glimpse of energy-critical theory: profile decomposition, perturbation theory

**Other particulars:**
- This course was offered as a joint MIGSAA and UoE level 10 course in Spring 2016 over the period of 11 weeks (for 3 hours/week), where
  - The UoE students stayed for the first 2/3 of the course (counting it as a full UoE advanced undergraduate course), and
  - The MIGSAA students stayed in the course for the full 11 weeks.
This course will most likely be offered in Spring 2018 as a UoE course. Thus, if it is offered as an UoE course, it is most convenient to keep the same format from Spring 2016.
- This course had $2 \times 1.5$ hour classes/week in Spring 2016 and would like to keep the same format. After a brief review of basic materials, this course directly dives into a research level material. As such, assignments and exams are not so effective as those for lower level courses. Thus, I would prefer to have extra hours of lectures and count the attendance and typing lecture notes (filling in details) as course assessment.
Analysis of Diffusion Processes - Michela Ottobre (Tuesdays 9.00 am – 10.30 am)

Credits: 15

The theory of diffusion processes has a very rich mathematical structure. One of the key features of such a theory is the interplay between probabilistic and analytic techniques. Analytic techniques are employed to give a macroscopic description of the dynamics, while probabilistic tools (stochastic analysis and stochastic processes) are used for the microscopic description.

1. Lecturer: Michela Ottobre

2. Course duration and format: 20 hours (correspondingly, 15 credits); part of the course will be taught through classic lectures and part will involve active student participation (sort of reading group style)

3. The course would ideally take place during the second term of the next academic year, i.e. starting in January 2018.

4. Syllabus. The course will present the theory of time-homogeneous diffusion processes from the analysis standpoint. It will be split in two parts of roughly 10 hours each. Students who have attended the course in 2015 should skip the first part but they are very welcome to attend the second part (and in this case they would be awarded 7.5 credits). The first half of the course will include:

   • Markov Semigroups and their generators. Dual semigroup. Invariant and reversible measures
   • Ergodic Theory for Markov Semigroups: Strong Feller semigroups, Krylov-Bogoliubov Theorem, Doob’s Theorem, Prokhorov’s theorem
   • Diffusion processes: definition, relation with stochastic differential equations, Backward Kolmogorov and Fokker-Planck equation, Feynman-Kac
   • Reversible diffusions: spectral gap inequality, exponentially fast convergence to equilibrium
   • Over and under-dumped Langevin equation

The second half will focus on hypoelliptic diffusions; in particular

   • Elliptic and hypoelliptic diffusions
   • The Hoermander condition (HC): this is a sufficient condition for hypoellipticity and will be presented by a probabilistic, analytic and geometric standpoint. In this respect, after introducing the HC (which stems from purely analytic considerations) we will discuss
   – Some basic notions of differential geometry: vector fields, integral curves and distributions (distributions in geometrical sense, not in probabilistic sense), orbits of a vector field
   – Geometric meaning of the HC: Chow’s theorem and Sussman’s orbit theorem
   – Probabilistic bearings of the Hoermander condition: existence of a density for the law of SDEs
   (The three above points may seems unrelated; as it turns out, they are very closely related indeed!)
   • Examples in statistical mechanics: heat bath models, second order Langevin equation

5. Course Material. Lecture notes will be provided, containing an extensive bibliography. The lecture notes will be complemented with further reading material. An indicative (non-final) list of references is the following

   i) Analysis and Geometry of Markov Diffusion operators, by D. Bakry et al.
   ii) Second Order Partial Differential Equations in Hilbert Spaces, Da Prato and Zabczyk
   iii) To begin at the beginning... D. Williams
   iv) Analytical Methods for Markov Semigroups. L. Lorenzi and M. Bertoldi
   v) Hypoelliptic Second order differential equations. L. Hoermander
   ii) On Malliavin’s proof of Hoermander’s theorem. M. Hairer

Further and more detailed references will be given during the course.

6. Relation to other courses. The background for this course is given by the Probability 1 SMSTC stream
(and some lectures of the Probability 2 stream), where basic stochastic calculus and the basic theory of Markov Processes is covered. Moreover students with a background in analysis will find obvious relations with the theory of parabolic PDEs. There are strong links with the courses on dissipative PDEs as well.

7. **Prerequisites.** Basic probability theory, basic stochastic calculus (e.g. Itô formula), very basic SDE and PDE theory

**What is ... Numerical Analysis? - Heiko Gimperlein (Tuesdays 10.45 am – 12.45 pm)**

Format: 10 lectures of 2 hours each, additional student lectures A similar course "What is ... PDEs?" currently runs informally at HW. The format is highly interactive, where students and upcoming seminar talks determine the content of the lectures.

Credits: 15 for students giving a lecture, less for participation or active contribution to tutorials

Aim: This course aims to give an introduction to standard techniques in the numerical analysis of partial differential equations, with a focus on the underlying analysis.

Prerequisites: a previous course in either PDE or their numerical analysis

Contents:

We cover some essential basic and advanced topics in the numerical analysis of PDEs. After the course the student should know key ideas in a broad range of topics, as they are relevant in their research or in relevant numerical analysis talks.

In particular, we expect to touch on the following topics:

* Basics I: Numerical methods, such as finite differences, finite elements, finite volume methods, boundary elements, time-stepping schemes
* Basics II: Relevant topics in analysis, such as approximation properties of functions, Sobolev spaces and functional analysis
* Finite element methods for elliptic problems: Conforming variational and mixed methods, error analysis, adaptive methods
* Non-conforming and non-standard methods
* Finite elements for the Stokes problem, analysis and stabilisation
* Heat and wave equations: time-stepping schemes and their analysis
* Fast solvers: review of numerical linear algebra, preconditioning, multigrid methods
* Applications in computational mechanics, fluid dynamics or biology

Some references:
* D. Braess, Finite elements: Theory, fast solvers, and applications in solid mechanics, Cambridge University Press
* H. Gimperlein, Interface and contact problems, lecture notes
* Y. Saad, Iterative methods for sparse linear systems, SIAM
* E.P. Stephan, Theory of approximation methods, lecture notes

Student talks:
Interested students will give a 60-minute lecture on a topic of their choice, ideally a topic related to their research interests.

**Advanced PDE II: Hyperbolic PDEs -- Pieter Blue and Oana Pocovnicu (Wednesdays 9 – 10.50 am)**

**Credits:** (15 credits)

Brief description: This course is dedicated to the study of hyperbolic PDEs in Sobolev spaces. We will mainly focus on nonlinear wave equations and symmetric hyperbolic systems. The course will begin with some preliminary ideas including the method of characteristics, finite speed of propagation, and finite time blow-up. Then, there will be a brief discussion of classical methods including the explicit solution formula of D'Alembert and Kirchhoff and the Cauchy-Kovalevskaya existence and uniqueness theorem. The course will
then turn to Sobolev space methods. These include energy estimates, Klainerman-Sobolev inequality, the vector-field method, and well-posedness of semilinear wave equations using Sobolev estimates. In the final two weeks we will cover special topics at the instructor’s discretion. Possible special topics are small data global well-posedness of quasi-linear wave or Klein-Gordon equations (study started independently in works of Klainerman and Shatah from the 1980s), the global well-posedness of the defocusing energy-critical semilinear wave equation (work of Shatah-Struwe, 1994), or a survey of geometric hyperbolic PDE such as the Yang-Mills and wave maps equations.

Prerequisites: Students should be familiar with Banach and Hilbert spaces, dual spaces, and weak and strong convergence. The course “Advanced PDEs I” is suggested but not required.

Homogenisation I - Mariya Ptashnyk (Thursdays 11.15 am – 12.45 pm)

Credits: 15
Course Hours: 20

Homogenization theory: multiscale modelling and analysis of physical and biological processes

The aim of homogenization theory is to determine the macroscopic behaviour of a system comprising microscopic heterogeneities, e.g. transport processes in a porous medium, signalling processes in a cell tissue, deformations of composite materials. This means that the mathematical model defined in a heterogeneous medium is replaced by equations posed in a homogeneous one, which provide a good approximation of properties of the original microscopic system.

In this course we will learn the main methods of homogenization theory, which are used to prove that solutions of microscopic problem, depending on a small parameter, converge to a solution of the corresponding macroscopic problem, as the small parameter (determined by the characteristic size of the microscopic structure) goes to zero.

Syllabus:
• main methods of periodic homogenization: formal asymptotic expansion, two-scale convergence, unfolding operator
• derivation of compactness results for two-scale convergence and periodic unfolding operator
• multiscale modelling and analysis of fluid flow in porous media
• multiscale modelling and analysis of transport and reaction processes in perforated and partially perforated domains (i.e. signalling and transport processes in biological tissues, plant root growth)
• dual-porosity: modelling and multiscale analysis (transport and reaction processes in fractured media, in soil with porous particles, in cell tissues)
• multiscale analysis of equations of linear elasticity and viscoelasticity
• main ideas of Gamma-, G- and H- convergences

Selected references:
Cioranescu D., Donato P. An introduction to homogenization, Oxford University Press, 1999
Braides A. Gamma-Convergence for beginners, Oxford University Press, 2002
Braides A. A handbook of Gamma-convergence, online

Assessment: (20 hrs, 15 credits)
7 credits for active participation
8 credits for solving tutorial/homework questions (under guidance of instructor)

Recommended prerequisites: some knowledge of Sobolev spaces and PDEs.
In this course, various concepts on stochastic homogenization theory are introduced with a view on multiscale modelling of multiphase systems. Stochastic homogenisation is a reliable and systematic theory for averaging partial differential equations defined on strongly heterogeneous media and domains with random characteristics. It has a wide range of applications from (reactive) transport in porous media, over to waves in heterogeneous media up to material science such as energy storage/conversion devices. It allows to rigorously derive effective macroscopic properties of strongly heterogeneous random media such as composite materials, the effective macroscopic formulation of microscopic systems, as well as the stable construction of multiscale computational schemes. We shall illustrate these features by considering various examples from continuum mechanics and physics of composite materials and porous media.

Structure of the course:

We begin with deriving effective stochastic differential equations (SDEs) based on the asymptotic two-scale expansion method. This provides a systematic tool to derive effective diffusion coefficients for heterogeneous materials. We will briefly discuss how numerically solve SDEs and their effective/upscaled formulation. We then introduce the two-scale convergence methodology which is the basis for the stochastic two-scale convergence and the stochastic two-scale convergence in the mean. As before, we will discuss how to compute a numerical approximation of the resulting limit problems. The last topic of the course will be on general concepts of percolation theory and investigate similarities and differences to homogenization. Again, we will also give the basic ideas how to computationally study percolation problems. All these topics will be discussed based on examples and applications such as interacting particle systems under uncertainty/randomness, and if time allows we look also at the theory of fluctuations and correlations.

Assessment: (20 hrs give 15 credits)

- 7 credits for active participation
- 8 credits (for worked out lecture notes under guidance of instructor)

Recommended but optional prerequisites:

This course is a follow up on the course HOMOGENIZATION, but students and researchers interested in stochastic averaging techniques will be able to follow it without having attended the first course. The following experience is helpful but not required: Advanced PDEs 1 and basic knowledge in Probability Theory and stochastic differential equations and associated Kolmogorov equations. Useful are also basic knowledge about Measure and Integration Theory, and Functional Analysis. Basic knowledge about Galerkin/Finite Element approximations and Finite Difference methods.

Syllabus:

- general concepts of stochastic differential equations and percolation theory; two-scale convergence, stochastic two-scale convergence, and stochastic two-scale convergence in the mean for elliptic PDEs; stochastic modelling of a hard-sphere interacting particle system; finite different approximations of homogenized equations; finite element discretizations of the stochastic two-scale limit;

Selection of references:


Advanced Mini Courses

From time to time we put on short courses of varying length (and with varying credit) which are typically given by visiting researchers. We will keep you posted when such courses become available.

Other courses in the pool -- which we may mount according to interest. (Further suggestions are welcome at any time.)

• Advanced mechanics
• Stochastic analysis
• Modelling and computation
• Advanced topics in elliptic and parabolic nonlinear PDE
• Weak convergence methods for nonlinear PDE
• Variational methods
• Toeplitz operators and applications
• Singular and oscillatory integrals and geometric applications
• Harmonic analysis methods for PDE
• Probabilistic methods for dispersive PDE
• The Cauchy problem in general relativity
• Harmonic analysis and discrete geometry
• Stability of stochastic processes
• Rare events and heavy tails, with applications
• L2 and Lp theory for PDE
• Mathematical analysis and large data sets
• Harmonic analysis methods in number theory
• Topological methods in analysis
• Analytical methods in modern informatics
• Low-regularity theory for dispersive equations
• Numerical methods for SDE
• Density functional theory and applications
• Finite element methods

Note that only a very small selection of these will run at any one time and most advanced modules will be offered at most once every other year.

Isabelle Hanlon 14/09/17